

# Dispersionless integrable equations as coisotropic deformations. Extensions and reductions

B.G. Konopelchenko<sup>1</sup> and F. Magri<sup>2</sup>

<sup>1</sup>Dipartimento di Fisica, Università di Lecce and Sezione INFN,  
73100 Lecce, Italy

<sup>2</sup>Dipartimento di Matematica ed Applicazioni, Università di Milano Bicocca,  
20126 Milano, Italy

## Abstract

Interpretation of dispersionless integrable hierarchies as equations of coisotropic deformations for certain associative algebras and other algebraic structures is discussed. It is shown that within this approach the dispersionless Hirota equations for dKP hierarchy are nothing but the associativity conditions in a certain parametrization. Several generalizations are considered. It is demonstrated that the dispersionless integrable hierarchies of B type like the dBKP hierarchy and the dVN hierarchy represent themselves the coisotropic deformations of the Jordan's triple systems. Stationary reductions of the dispersionless integrable equations are shown to be connected with the dynamical systems on the plane completely integrable on a fixed energy level.

# 1 Introduction to the concept of coisotropic deformations.

Dispersionless integrable equations and hierarchies have attracted considerable interest during the last two decades (see e.g. [1-16]). A principal example of such equations is given by the dispersionless Kadomtsev-Petviashvili (dKP) equation

$$\begin{aligned}\frac{\partial u}{\partial x_3} &= \frac{3}{2}u \frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_2}, \\ \frac{\partial v}{\partial x_1} &= \frac{3}{4} \frac{\partial u}{\partial x_2}.\end{aligned}\tag{1.1}$$

This equation is equivalent to the compatibility condition

$$\frac{\partial^2 S}{\partial x_2 \partial x_3} = \frac{\partial^2 S}{\partial x_3 \partial x_2}$$

for the pair of equations

$$\frac{\partial S}{\partial x_2} = \left( \frac{\partial S}{\partial x_1} \right)^2 + u,\tag{1.2}$$

$$\frac{\partial S}{\partial x_3} = \left( \frac{\partial S}{\partial x_1} \right)^3 + \frac{3}{2}u \frac{\partial S}{\partial x_1} + v.\tag{1.3}$$

The higher dKP equations arise as the compatibility conditions for equation (1.2) and equations (see [1]-[10])

$$\frac{\partial S}{\partial x_n} = \left( \frac{\partial S}{\partial x_1} \right)^n + \sum_{m=0}^{n-2} u_{nm}(x) \left( \frac{\partial S}{\partial x_1} \right)^m.\tag{1.4}$$

Other dispersionless integrable equations are usually associated with the Hamilton-Jacobi type equations of the form [9, 10]

$$\frac{\partial S}{\partial x_n} = \Omega_n \left( \frac{\partial S}{\partial x_1}, x \right)\tag{1.5}$$

where  $\Omega_n$  are meromorphic functions of the first argument.

In this standard formulation the variables  $x_1$  and  $p = \frac{\partial S}{\partial x_1}$  play a distinguished role. They form the pair of canonically conjugate variables and the compatibility conditions for the equations (1.5) can be written as

$$\frac{\partial \Omega_n}{\partial x_m}(p, x) - \frac{\partial \Omega_m}{\partial x_n}(p, x) - \{\Omega_n(p, x), \Omega_m(p, x)\} = 0\tag{1.6}$$

where the Poisson bracket  $\{\cdot, \cdot\}$  is defined as

$$\{f, g\} = \frac{\partial f}{\partial x_1} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial x_1}.\tag{1.7}$$

So the whole standard approach deals with the nonstationary Hamilton-Jacobi equations (1.5) and the Hamiltonians  $\Omega_n(p, x)$ .

In the present paper we will consider a novel approach to the dispersionless integrable equations and hierarchies in which they represent themselves the coisotropic deformations of associative algebras or other algebraic structures [17].

This approach which is a melting of ideas borrowed from Hamiltonian mechanics and theory of associative algebras starts with the reinterpretation of the Hamilton-Jacobi equations (1.5) as the stationary ones. Within such a viewpoint it is natural to consider all variables  $x_1, x_2, x_3, \dots$  on the equal footing and to introduce the corresponding canonical variables  $p_1, p_2, p_3, \dots$  conjugate to  $x_1, x_2, x_3, \dots$  with respect to the standard Poisson bracket  $\{\cdot, \cdot\}$  given by

$$\{f, g\} = \sum_{i=1}^n \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x_i} \right). \quad (1.8)$$

Thus we introduce the symplectic manifold  $M^{2n}$  equipped with the Poisson bracket (1.8). Hamilton-Jacobi equations (1.5) are substituted by the zero set of Hamiltonians

$$h_n \equiv -p_n + \Omega_n(p_1, x) = 0. \quad (1.9)$$

We emphasize that now  $x_i, p_i$  form pairs of canonical conjugate variables and  $p_i \neq \frac{\partial S}{\partial x_i}$ . We will denote the submanifold  $M^{2n}$  given by zero set (1.9) as  $\Gamma$ .

For instance, instead of equations (1.2), (1.3) one now has the zero set of the Hamiltonians

$$\begin{aligned} h_2 &= -p_2 + p_1^2 + u(x_1, x_2, x_3), \\ h_3 &= -p_3 + p_1^3 + \frac{3}{2}up_1 + v. \end{aligned} \quad (1.10)$$

The submanifold  $\Gamma$  defined by the equations (1.10) is “parametrized” by the functions  $u$  and  $v$  of the coordinates  $x_1, x_2, x_3$ .

The second step in our approach is to introduce a special class of submanifolds  $\Gamma$ . It is quite natural to require that Hamiltonian flows generated by the Hamiltonians  $h_2$  and  $h_3$  preserve  $\Gamma$  or, in other words, the Hamiltonian vector fields associated with  $h_2$  and  $h_3$  are tangent to  $\Gamma$ . This requirement is equivalent to the condition

$$\{h_2, h_3\}|_{\Gamma} = 0 \quad (1.11)$$

It is easy to check that this condition is satisfied if  $u$  and  $v$  obey the dKP equation (1.1). In this case  $\{h_2, h_3\} = 0$ .

Thus, for any solution  $u, v$  of the dKP equation (1.1) the Hamiltonians  $h_2$  and  $h_3$  are in involution and the deformation of  $x_1, x_2, x_3$  according to the Hamiltonian flows preserve  $\Gamma$ .

We will refer to such deformations as the coisotropic deformations due to the following reason. The submanifold  $\Gamma$  defined by the equations (1.10) and obeying the above conditions is the 4-dimensional submanifold in  $M^6$  for

which  $\{h_2, h_3\}|_\Gamma = 0$  and moreover the restriction of the standard symplectic 2-form

$$\omega \equiv \sum_{i=1}^3 dp_i \wedge dx_i \quad (1.12)$$

on  $\Gamma$  does not vanish. Since the Hamiltonian vector fields associated with  $h_2$  and  $h_3$  span the kernel of  $\omega$  then the rank of  $\omega|_\Gamma$  is equal to two and, hence,

$$\omega_\Gamma = d\mathcal{L} \wedge d\mathcal{M} \quad (1.13)$$

where  $\mathcal{L}$  and  $\mathcal{M}$  are canonical variables on  $\Gamma$ . The submanifolds with such properties are called coisotropic submanifolds (see e.g. [18, 19]).

So, any solution  $u, v$  of the dKP equation defines through (1.10) a 4-dimensional coisotropic submanifold in  $M^6$  and it is then natural to refer to the associated deformations as coisotropic ones.

Similarly, the conditions  $\{h_n, h_m\} = 0$  for the Hamiltonians  $h_n$  given by (1.9) define coisotropic deformations. One can easily check that the coisotropy conditions for Hamiltonians (1.9) exactly coincides with (1.6). In particular, for polynomials  $\Omega_n(p_1)$  of the type (1.10) the coisotropic deformations are given by the dKP hierarchy.

Reformulation of the dispersionless integrable equations and hierarchies as the coisotropic deformations not only gives us a technical simplification but also opens a way for essential generalizations and a novel interpretation [17].

A way for considerable extension of the above approach lies in the observation that one can pass from the zero set of the functions  $h_n$  to the ideal  $J$  generated by them. Indeed, any polynomial function  $f_n$  of  $h_1, h_2, \dots$  belong to  $\Gamma$  and the coisotropy condition  $\{f_n, f_m\}|_\Gamma = 0$  is satisfied due to  $\{h_n, h_m\}|_\Gamma = 0$ . Moreover, the both sets of functions give rise to the same differential equations for the potentials.

So the coisotropy condition takes the form of the closeness of  $J$  with respect to the Poisson bracket (1.8), i.e.

$$\{J, J\} \subset J. \quad (1.14)$$

A passage to the ideal  $J$  gives us the freedom to choose its basis in different ways. Let us consider the ideal generated by the dKP Hamiltonians  $h_0 \equiv -p_0 + 1$  and

$$h_n = -p_n + p^n + \sum_{m=0}^{n-2} u_{nm}(x)p^m, \quad n = 1, 2, 3, \dots \quad (1.15)$$

as an example, where for convenience we add Hamiltonians  $h_0 = -p_0 + 1$  and  $h_1 = -p_1 + p$ . Formally, we have to introduce the variable  $x$  conjugate to  $p$ . This implies that  $u_{nm}$  will depend only on  $x_1 + x$ . First, since  $-p_2 + p_1^2 + u = 0$  on  $\Gamma$  one has  $u = p_2 - p_1^2$ . Substituting this expression for  $u$  into  $h_3$ , one gets a new Hamiltonian

$$\tilde{h}_3 = -p_3 - \frac{1}{2}p_1^3 + \frac{3}{2}p_1p_2 + v. \quad (1.16)$$

Continuing such procedure, one obtains an infinite set of Hamiltonians  $\tilde{h}_n$  which have a form of the sum of polynomials in  $p_1, p_2, \dots$  and the functions  $u_n(x_1, x_2, \dots)$ , namely,

$$\tilde{h}_n = nP_n(\tilde{p}) + u_n(x), \quad n = 2, 3, \dots \quad (1.17)$$

where  $\tilde{p} = (-p_1, -\frac{1}{2}p_2, -\frac{1}{3}p_3, \dots)$ ,  $u_2 = u_{20} = u$ ,  $u_3 = u_{30} = v$  and  $P_n(t)$  are Schur's polynomials defined by the formula  $\exp(\sum_{n=1}^{\infty} z^n t_n) = \sum_{m=0}^{\infty} z^m P_m(t)$ . The Hamiltonians  $\tilde{h}_n$  together with  $h_0$  and  $h_1$  form a basis for the same ideal  $J$  generated by the functions  $h_n$  (1.15).

For the Hamiltonians (1.17) the coisotropy condition (1.14) implies that

$$\{\tilde{h}_n, \tilde{h}_m\} = \sum_{k=1}^{n-2} \frac{n}{k(n-k)} \frac{\partial u_m}{\partial x_k} \tilde{h}_{n-k} - \sum_{k=1}^{m-2} \frac{m}{k(m-k)} \frac{\partial u_n}{\partial x_k} \tilde{h}_{m-k} \quad (1.18)$$

and the following equations are satisfied

$$\frac{n-1}{n} \frac{\partial u_n}{\partial x_{m-1}} = \frac{m-1}{m} \frac{\partial u_m}{\partial x_{n-1}}, \quad (1.19)$$

$$\frac{\partial u_m}{\partial x_n} - \frac{\partial u_n}{\partial x_m} + \sum_{k=1}^{m-2} \frac{m}{k(m-k)} u_{m-k} \frac{\partial u_n}{\partial x_k} - \sum_{k=1}^{n-2} \frac{n}{k(n-k)} u_{n-k} \frac{\partial u_m}{\partial x_k} = 0 \quad (1.20)$$

where  $m, n = 2, 3, \dots$ . One can check that (1.19) and (1.20) are equivalent to the standard dKP hierarchy.

An interesting basis arises if one shall try to convert the basis (1.15) into that bilinear in  $p_1, p_2, p_3, \dots$ . Such a basis can be easily built in the following way. From the first equation (1.10) one has  $p_1^2 = p_2 - u$ . Substituting this expression for  $p_1^2$  into the second equation (1.10), one gets the equation

$$-f_{12} \equiv p_1 p_2 - p_3 + \frac{1}{2} u p_1 + v = 0. \quad (1.21)$$

Then, getting from the second equation (1.10) the expression  $p_1^3 = p_3 - \frac{3}{2} u p_1 - v$  and substituting it and  $p_1^2 = p_2 - u$  into equation (1.15) with  $n = 4$ , one arrives at the equation

$$-f_{13} = p_1 p_3 - p_4 + \left(u_{42} - \frac{3}{2} u\right) p_2 - v p_1 + u_{40} - u u_{42} = 0. \quad (1.22)$$

Repeating such a procedure, one gets the family of Hamiltonians of the form [17]

$$-f_{jk} \equiv p_j p_k - \sum_{l=0}^{j+k} C_{jk}^l(x) p_l = 0, \quad j, k = 1, 2, 3, \dots \quad (1.23)$$

where  $C_{jk}^l$  are certain coefficients and we denote  $p_0 \equiv 1$ . The minus sign in the l.h.s. of the above formulas is chosen in order to have the same definition of  $f_{jk}$  as that in [17]. The relation

$$f_{jk} = h_j h_k - \sum_{l \geq 0} C_{jk}^l(x) h_l + p_j h_k + p_k h_j$$

shows that the zero set of  $h_k$  coincides with zero set of  $f_{jk}$ . Direct but cumbersome calculations demonstrate that the coisotropy conditions for the Hamiltonians  $f_{jk}$  again give rise to the dKP hierarchy.

So, the choice of the basis in the ideal  $J$  does not affect the coisotropy conditions. But, remarkably, the above freedom allows us to reveal a deep algebraic roots of the dispersionless integrable equations. Indeed, equations (1.23) look very much like the realization of the table of multiplication for an associative algebra in the basis formed by  $p_0, p_1, p_2, \dots$ . This observation leads us to the possibility to treat the dKP hierarchy as the coisotropic deformations of a commutative associative algebra [17].

Namely, let us have a commutative associative algebra. We choose a basis  $p_0 (\equiv 1), p_1, p_2, \dots$  in the algebra and so we have the corresponding table of multiplication

$$p_j p_k = \sum_{l=0} C_{jk}^l p_l. \quad (1.24)$$

The commutativity means that  $C_{jk}^l = C_{kj}^l$  while the associativity implies

$$\sum_{l=0} C_{jk}^l C_{lm}^p = \sum_{l=0} C_{mk}^l C_{lj}^p. \quad (1.25)$$

To consider deformations we assume that the structure constants  $C_{jk}^k$  depend on a set of variables  $x_1, x_2, x_3, \dots$ . To specify deformations we associate with the table of multiplication (1.24) the set of quadratic Hamiltonians

$$f_{jk} = -p_j p_k + \sum_{l=0} C_{jk}^l(x) p_l, \quad (1.26)$$

where  $x_j, p_j$  form pairs of canonically conjugate variables in the symplectic manifold  $M$  equipped with the Poisson bracket (1.8).

Then we consider the polynomial ideal  $J = \langle f_{jk} \rangle$  generated by these Hamiltonians and the submanifold  $\Gamma$  defined by

$$\Gamma = \{(x_j, p_j \in M; f_{jk} = 0)\}. \quad (1.27)$$

Finally, if the ideal  $J$  is closed with respect to the Poisson bracket, i.e.

$$\{J, J\} \subset J \quad (1.28)$$

so that  $\Gamma$  is a coisotropic submanifold of  $M$ , the functions  $C_{jk}^l(x)$  are said to define a coisotropic deformation of the associative algebra defined by (1.24) [17].

One can show that the condition (1.28) is satisfied if the structure constants  $C_{jk}^l$  obey the system of equations

$$\sum_{s=1} \left( C_{sj}^m \frac{\partial C_{lr}^s}{\partial x_k} + C_{sk}^m \frac{\partial C_{lr}^s}{\partial x_j} - C_{sr}^m \frac{\partial C_{jk}^s}{\partial x_l} - C_{sl}^m \frac{\partial C_{jk}^s}{\partial x_r} + \frac{\partial C_{jk}^m}{\partial x_s} C_{jr}^s - \frac{\partial C_{lr}^m}{\partial x_s} C_{jk}^s \right) = 0 \quad (1.29)$$

The system of equations (1.25) and (1.29) completely defines coisotropic deformations of the associative algebra (1.24) and it is a basic one in our approach [17]. For this reason we will refer to it as the central system of the theory of coisotropic deformations.

We emphasize that the central system is the system of algebraic and differential equations for the structure constants  $C_{jk}^l$  only. We shall see that this feature is an essential advantage of the approach which we discuss here.

## 2 dKP coisotropic deformations: tau function and Hirota's equations.

A central problem concerning the central system (1.25), (1.29) is, of course, the existence and meaning of its nontrivial solutions. A solution is provided by the dKP hierarchy for which the structure constants  $C_{jk}^l$  can be reconstructed through the potentials  $u_{nm}(x)$  in (1.15) via the formula (1.23).

On the other hand an analysis of equations (1.23) shows [17] that the dKP structure constants  $C_{jk}^l$  have the following general form

$$C_{kj}^l = \delta_{k+j}^l + H_{j-l}^k + H_{k-l}^j \quad (2.1)$$

where  $k, j, l = 0, 1, 2, 3, \dots$ ,  $\delta_k^l$  is the Kroneker symbol and  $H_j^k$  are functions of  $x_1, x_2, \dots$  such that  $H_0^k = 0$  and  $H_j^k = 0$  for  $j \leq -1$ . We will show that the use of the general (unparametrized) form (2.1) of the structure constants in the central system will lead us to the existence of the tau-function and a novel algebraic interpretation of the dispersionless Hirota's equations [17].

The first step is to implement the associativity conditions on the coefficients  $H_j^k$ . Direct substitution of expressions (2.1) into equations (1.15) and the use of the identity

$$\sum_{l=n-1}^{k-1} H_{k-l}^i H_{l-n}^m = \sum_{l=n-1}^{k-1} H_{k-l}^m H_{l-n}^i,$$

shows that the structure constants  $C_{jk}^l$  obey the associative conditions if and only if the bracket

$$[H, H]_{ikm} := H_m^{i+k} - H_{m+k}^i - H_{i+m}^k + \sum_{l=1}^{i-1} H_{i-l}^k H_m^l + \sum_{l=1}^{k-1} H_{k-l}^i H_m^l - \sum_{l=1}^{m-1} H_{m-l}^k H_l^i \quad (2.2)$$

vanishes identically for any choice of the indices  $(i, k, m) \in N$ . An interesting consequence of this result can be drawn by contraction. Indeed one may check that the above equations imply the useful symmetry relations

$$pH_p^i = iH_i^p. \quad (2.3)$$

Next, one has to implement the coisotropy conditions. In terms of the coefficients  $H_k^i$  the ensuing equations are rather complicated. However a closer

scrutiny shows that they are simplified drastically on account of the associativity conditions just obtained. Indeed one can prove that the coisotropy conditions may be reduced to the equations

$$\frac{\partial[H, H]_{l,n,i-j}}{\partial x_k} + \frac{\partial[H, H]_{l,n,k-j}}{\partial x_i} - \frac{\partial[H, H]_{i,k,l-j}}{\partial x_n} - \frac{\partial[H, H]_{i,k,n-j}}{\partial x_l} = 0,$$

which are automatically fulfilled owing to the associativity conditions, and to the *linear* equations

$$\frac{\partial H_p^i}{\partial x_l} = \frac{\partial H_p^l}{\partial x_i}.$$

So, to summarize, the associativity and coisotropy conditions in the case of the structure constants of the form (2.1) are together equivalent to the set of quadratic algebraic equations

$$[H, H]_{ikl} = 0, \quad (2.4)$$

and to the set of linear differential equations

$$\frac{\partial H_p^i}{\partial x_l} = \frac{\partial H_p^l}{\partial x_i} \quad (2.5)$$

having the form of a system of conservation laws. The equations (2.4) and (2.5) give the specific form of the *central system* for the dKP hierarchy. It encodes all the informations about the hierarchy. In particular it entails that for any solution of the central system one has

$$\{f_{ik}, f_{ln}\} = \sum_{s,t \geq 1} K_{ikln}^{st} f_{st} \quad (2.6)$$

where

$$K_{ikln}^{st} = \left( \delta_{it} \frac{\partial}{\partial x_k} + \delta_{kt} \frac{\partial}{\partial x_i} \right) (H_{l-s}^n + H_{n-s}^l) - (\delta_{nt} \frac{\partial}{\partial x_l} + \delta_{lt} \frac{\partial}{\partial x_n}) (H_{i-s}^k + H_{k-s}^i).$$

From this formula one sees that the Hamiltonians  $f_{jk}$  of the dispersionless KP hierarchy form a Poisson algebra. The above central system can be seen also as the dispersionless limit of the central system of the full dispersive KP hierarchy [20].

There are presently two strategies to decode the informations contained in the central system. According to the first strategy, one first tackles the associativity conditions (2.4), noticing that they allow to compute the coefficients  $(H_k^2, H_k^3, \dots)$  as polynomial functions of  $H_k^1$ . For instance, the symmetry conditions

$$H_1^2 = 2H_2^1 \quad H_1^3 = 3H_3^1$$

give  $(H_1^2, H_1^3)$ , and then the condition (2.4) with  $i = k = 1, l = 2$ , i.e.

$$H_1^3 - H_3^1 - H_2^2 + H_1^1 H_1^1 = 0$$



gives  $H_2^2$ , and so forth. Renaming the free coefficients as suggested by the table of multiplication of the previous section, that is by setting

$$H_1^1 = -1/2u, \quad H_2^1 = -1/3v, \quad H_3^1 = -1/4w + 1/8u^2,$$

one gets

$$H_1^2 = -2/3v, \quad H_2^2 = -1/2w + 1/2u^2, \quad H_1^3 = -3/4w + 3/8u^2.$$

At this point one plugs these expressions into the simplest linear coisotropy conditions

$$\frac{\partial H_1^1}{\partial x_2} - \frac{\partial H_1^2}{\partial x_1} = 0, \quad \frac{\partial H_2^1}{\partial x_2} - \frac{\partial H_2^2}{\partial x_1} = 0, \quad \frac{\partial H_1^1}{\partial x_3} - \frac{\partial H_1^3}{\partial x_1} = 0,$$

arriving to the equations

$$\begin{aligned} \frac{\partial v}{\partial x_1} &= 3/4 \frac{\partial u}{\partial x_2}, \\ \frac{\partial v}{\partial x_2} &= 3/2 \frac{\partial w}{\partial x_1} - 3u \frac{\partial u}{\partial x_1}, \\ \frac{\partial u}{\partial x_3} &= 3/2 \frac{\partial w}{\partial x_1} - 3/2u \frac{\partial u}{\partial x_1}. \end{aligned}$$

The elimination of  $\frac{\partial w}{\partial x_1}$  leads finally to the dKP equation and to the higher equations, if one insists enough in the computations. By this strategy one come back to the hierarchy in its standard formulation.

A simple inversion of the order in which the equations are considered leads instead to the Hirota's formulation. It is enough to remark that equations (2.5) entail the existence of a sequence of potentials  $S_m$  such that

$$H_m^i = \frac{\partial S_m}{\partial x_i}.$$

Then the symmetry conditions (2.3), oblige the potentials  $S_m$  to obey the constraints

$${}_i \frac{\partial S_i}{\partial x_l} = {}_l \frac{\partial S_l}{\partial x_i},$$

which in turn entail the existence of a superpotential  $F(x_1, x_2, \dots)$  such that

$$S_i = -1/i \frac{\partial F}{\partial x_i}, \quad H_m^i = -1/m \frac{\partial^2 F}{\partial x_i \partial x_m}. \quad (2.7)$$

This result provides a second parametrization of the structure constants, after that described before. The insertion of the new parametrization into the full set

of associativity conditions finally leads to the system of equations

$$\begin{aligned}
& -\frac{1}{m}F_{i+k,m} + \frac{1}{m+k}F_{i,k+m} + \frac{1}{i+m}F_{k,i+m} \\
& + \sum_{l=1}^{i-1} \frac{1}{m(i-l)}F_{k,i-l}F_{l,m} + \sum_{l=1}^{k-1} \frac{1}{m(k-l)}F_{i,k-l}F_{l,m} \\
& - \sum_{l=1}^{m-1} \frac{1}{i(m-l)}F_{k,m-l}F_{i,l} = 0,
\end{aligned} \tag{2.8}$$

where  $F_{i,k}$  stands for the second-order derivative of  $F$  with respect to  $x_i$  and  $x_k$ . These equations are equivalent to the celebrated Hirota's bilinear equations for the tau function of the dispersionless KP hierarchy (see e.g. [7, 8, 14]). For instance, for  $(i = k = 1, m = 2)$  or  $(i = m = 1, k = 2)$  or  $(m = k = 1, i = 2)$  one obtains directly the first Hirota's equation

$$-1/2F_{2,2} + 2/3F_{1,3} - (F_{1,1})^2 = 0.$$

For  $(i = 1, k = 2, m = 2)$  or  $(i = 2, k = 1, m = 2)$  or  $(i = 1, k = 1, m = 3)$  it gives instead the second Hirota's equation

$$1/2F_{1,4} - 1/3F_{2,3} - F_{1,1}F_{1,2} = 0.$$

Higher equations (2.8) do not separately coincide with Hirota's bilinear equations, but are together equivalent to standard Hirota's equations of the same weight.

Thus, the dispersionless Hirota's equations for the dKP hierarchy is nothing but the associativity conditions (2.4) under the parametrization (2.7). We emphasize that this result is due to the use of the structure constants (2.1) in the central system instead of potentials  $u_{nm}$ .

We note that the functions  $K_{ikln}^{st}$  in the formula (2.6) are linear combinations of third- order derivatives of the tau function  $F$ . For example,

$$\{f_{11}, f_{1n}\} = -2 \sum_{s=1}^{s=n-1} \left( \frac{1}{(n-s)} \frac{\partial^3 F}{\partial x_1^2 \partial x_{n-s}} \right) f_{1s} \quad n = 2, 3, \dots \tag{2.9}$$

It is, probably, not just a coincidence that the third order derivatives of the tau function appear both in WDVV equations [21, 22, 14] and as the structure constants in equations (2.9).

We note also that the coisotropy conditions (2.9) break at the points where the third order derivatives of  $F$  blow up. These points correspond to the singular sector of the dKP hierarchy.

### 3 Extensions: dmKP, Harry Dym and d2DTL hierarchies.

Several generalizations of the results presented in the section 2 are associated with rather obvious extensions of the polynomial algebra defined by the for-

mula (1.15).

First extension is to relax the polynomials in the r.h.s of (1.15) permitting them to contain the terms  $u_{n,n-1}(x)p^{n-1}$ , i.e. to consider the family of Hamiltonians

$$\begin{aligned} h_0 &= -p_0 + 1 \\ h_1 &= -p_1 + p \\ &\dots \\ h_n &= -p_n + \sum_{m=0}^n u_{nm}(x)p^m, \quad n = 2, 3, 4, \dots \end{aligned} \quad (3.1)$$

with  $u_{nn} = 1$ . The coisotropy condition again is given by the equation  $\{h_n, h_m\} = 0$  where the Poisson bracket is defined as

$$\{f, g\} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} + \sum_{i=1}^n \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x_i} \right). \quad (3.2)$$

The variable  $x_0$  conjugate to  $p_0$  is a cyclic one and we omit the corresponding term from (3.2). In virtue of the form of  $h_1$  the coefficients  $u_{nm}$  depend on the sum  $x + x_1$ .

The coisotropy conditions for the Hamiltonians (3.1) give rise to the generalised dKP hierarchy (see [17]). In the special gauge  $u_{20} = 0$ ,  $u_{30} = 0$  one gets the dmKP hierarchy [17].

Further obvious extension is to allow the coefficients  $u_{nn}$  depend on  $x$ . The corresponding deformations contain the dispersionless Harry Dym hierarchy [23] as a subclass. Indeed, with the choice

$$\begin{aligned} h_2 &= -p_2 + \rho^2 p_1^2, \\ h_3 &= -p_3 + \rho^3 p_1^3 + \mu \rho^2 p_1^2 \end{aligned} \quad (3.3)$$

one gets from the coisotropy condition the system

$$\begin{aligned} \frac{\partial(\rho^2)}{\partial x_3} &= \frac{\partial(\mu \rho^2)}{\partial x_2}, \\ \frac{\partial \mu}{\partial x_1} &= \frac{3}{2} \frac{\partial \rho}{\partial x_2} \end{aligned} \quad (3.4)$$

that is the dispersionless 2 + 1-dimensional Harry Dym equation [23].

In the above extensions one remains within the class of polynomial algebras with the single generator  $p$ . A generalization of the whole approach to polynomial algebras with  $n$  generators via the gluing process has been proposed in [17]. In the simplest case of two generators  $p$  and  $q$  it consists in adding to two polynomial algebras (3.1) the gluing relation

$$pq = ap + bq + c. \quad (3.5)$$

The relation (3.5) allows us to build the whole table of multiplications between  $p_i q_k$ . It was shown in [17] that the corresponding coisotropic deformations are given by the two-point dKP hierarchy which is equivalent to the Whitham universal hierarchy on the Riemann sphere with two punctures [9]. In particular, it contains the simplest  $2+1$ -dimensional Benney equation introduced in [10].

Here we would like to indicate a way to incorporate the dispersionless two-dimensional Toda lattice (d2DTL) in our scheme. For this purpose we consider two general copies of polynomial algebras and, hence, two families of Hamiltonians

$$\begin{aligned} h_0 &= -p_0 + 1 \\ h_1 &= -p_1 + ap + b \\ &\dots \\ h_n &= -p_n + \sum_{m=0}^n u_{nm}(x, y) p^m \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \tilde{h}_0 &= h_0 = -p_0 + 1 \\ \tilde{h}_1 &= -q_1 + \tilde{a}q + \tilde{b} \\ &\dots \\ \tilde{h}_n &= -q_n + \sum_{m=0}^n \tilde{u}_{nm}(x, y) q^m \end{aligned} \tag{3.7}$$

glued by the relation

$$f = pq - 1 = 0. \tag{3.8}$$

Here  $a, b, u_{nm}, \tilde{a}, \tilde{b}, \tilde{u}_{nm}$  are functions of the variables  $x, x_2, x_2, \dots; y, y_1, y_2, \dots$  canonically conjugate to  $p, p_1, p_2, \dots; q, q_1, q_2, \dots$ .

Let us analyze now the coisotropy condition for these Hamiltonians. It is not difficult to check that the use of the canonical Poisson bracket of the type (1.7), for instance, for the Hamiltonians  $h_1, \tilde{h}_1, f$  gives rise to a too strong constraints on  $a, b, \tilde{a}, \tilde{b}$  and consequently to trivial deformations.

If instead one uses a modified Poisson bracket defined as

$$\begin{aligned} \{f, g\} &= p \left( \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} \right) - q \left( \frac{\partial f}{\partial y} \frac{\partial g}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial g}{\partial y} \right) \\ &+ \sum_{i=1} \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x_i} + \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial y_i} \right) \end{aligned} \tag{3.9}$$

the situation changes drastically. Indeed, from the conditions  $\{h_1, f\}|_{\Gamma} = 0$  and  $\{\tilde{h}_1, f\}|_{\Gamma} = 0$  one obtains

$$\begin{aligned} a_x - a_y &= 0, & b_x - b_y &= 0, \\ \tilde{a}_x - \tilde{a}_y &= 0, & \tilde{b}_x - \tilde{b}_y &= 0, \end{aligned} \tag{3.10}$$

while the condition  $\left\{h_1, \tilde{h}_1\right\}\Big|_{\Gamma} = 0$  gives

$$\begin{aligned} a_{y_1} + a\tilde{b}_x &= 0, \\ \tilde{a}_{x_1} - \tilde{a}b_y &= 0, \\ \tilde{a}a_y + a\tilde{a}_x + b_{y_1} - \tilde{b}_{x_1} &= 0. \end{aligned} \tag{3.11}$$

At the particular gauge  $b = -a, \tilde{b} = -\tilde{a}$  the system (3.10), (3.11) implies that

$$\begin{aligned} a_{y_1} - a\tilde{a}_x &= 0, \\ \tilde{a}_{x_1} + \tilde{a}a_x &= 0. \end{aligned} \tag{3.12}$$

From this system one gets

$$\theta_{x_1 y_1} + (e^\theta)_{xx} = 0 \tag{3.13}$$

where  $\theta = \log(a\tilde{a})$ . It is the Boyer-Finley or dispersionless two-dimensional Toda lattice (d2DTL) equation (see e.g. [5, 8, 9]). Considering the coisotropy conditions for the higher Hamiltonians  $h_n, \tilde{h}_n$ , one will obtain the higher d2DTL equations and the whole d2DTL hierarchy.

The fact that in order to get d2DTL as coisotropic deformations of the polynomial algebras one has to consider the modified Poisson bracket (3.9) is of considerable importance. It indicates that the symplectic structure of the deformations should be consistent with the algebra to be deformed. Of course, one can easily transform the modified bracket (3.9) into a canonical one introducing the variable  $w$  such that  $p = \exp(w), q = \exp(-w)$ . But in terms of this variable the Hamiltonians  $h_n, \tilde{h}_n$  become the polynomials of exponentials  $\exp(w)$  and  $\exp(-w)$ . This phenomena and the first half of the bracket (3.9) (at  $q = 1/p$ ) are well-known for the d2DTL hierarchy (see e.g. [5, 8, 9]).

Similar situation of consistency of Poisson bracket and the form of Hamiltonians take place for dispersionless equations considered in [24].

## 4 Coisotropic deformations of the Jordan's triple systems: hierarchies of B type.

Quite different and interesting structures arise for "algebras" compatible with a discrete group, i.e. in the cases when discrete group acts nontrivially on a basis while the structure constants remain invariant. Here we will consider only a simple example of the group of simultaneous reflections of all elements of a basis  $p_i \rightarrow -p_i$ .

Let us begin with the one component case. A natural basis now is given by

odd powers  $p^{2i+1}$  and the analogs of the polynomials (1.15) are of the form

$$\begin{aligned} p_1 &= p, \\ p_3 &= p^3 + up, \\ p_5 &= p^5 + v_3 p^3 + v_1 p, \\ p_7 &= p^7 + w_5 p^5 + w_3 p^3 + w_1 p \end{aligned} \quad (4.1)$$

and so on. Such an algebra, obviously, is not closed under the usual product operation  $p_i p_k$ . But the relations

$$\begin{aligned} p_1^3 &= p_3 - up_1, \\ p_1^2 p_3 &= p_5 + (u - v_3) p_3 - [u(u - v_3) + v_1] p_1, \\ p_1 p_3^2 &= p_7 + (2u - w_5) p_5 + [u^2 - w_3 - v_3(2u - w_5)] p_3 \\ &\quad + [(2u - w_5)(v_3 u - v_1) - u(u^2 - w_3) - w_1] p_1, \\ p_1^2 p_5 &= p_7 + (v_3 - w_5) p_5 + [v_1 - w_3 - v_3(v_3 - w_5)] p_3 \\ &\quad + [(v_3 - w_5)(v_3 u - v_1) - u(v_1 - w_3) - w_1] p_1 \end{aligned} \quad (4.2)$$

and similar one suggest to consider the cubic operation  $p_i p_k p_l$ . So, the odd polynomials (4.1) provide us with an example of the “algebra” with the basis  $p_1, p_3, p_5, \dots$  closed under the commutative trilinear operation

$$p_i p_k p_l = \sum_{m=1}^{i+k+l} C_{ikl}^m p_m \quad (4.3)$$

where all indices take only odd integer values, and

$$\begin{aligned} C_{111}^3 &= 1, & C_{111}^1 &= -u, & C_{113}^5 &= 1, \\ C_{113}^3 &= u - v_3, & C_{113}^1 &= -[u(u - v_3) + v_1] \end{aligned}$$

and so on. Algebraic structures defined by the trilinear law (4.3) are known as the Jordan’s triple systems (see e.g. [25, 26]). The associativity for the Jordan’s triple system is defined as

$$(p_i p_k p_l) p_n p_m = p_i p_k (p_l p_n p_m) = p_i (p_k p_l p_n) p_m \quad (4.4)$$

that leads to the following “associativity” conditions

$$\begin{aligned} \sum_t C_{ikl}^t C_{tnm}^s &= \sum_t C_{lnm}^t C_{ikt}^s, \\ \sum_t C_{lnm}^t C_{ikt}^s &= \sum_t C_{kln}^t C_{imt}^s. \end{aligned} \quad (4.5)$$

One can apply the approach discussed above to define coisotropic deformations of Jordan’s triple systems. So, we convert the multiplication table (4.3) into to the zero set  $\Gamma$  of the Hamiltonians

$$f_{ikl} := -p_i p_k p_l + \sum_{m=1} C_{ikl}^m(x) p_m, \quad i, k, l = 1, 3, 5, \dots \quad (4.6)$$

and then demand the coisotropy of  $\Gamma$  with respect to the canonical Poisson bracket. One gets the coisotropy deformations equations for the structure constants  $C_{ikl}^m$  which are similar to the equations (1.29).

Here we will consider the simplest of them which arise as the coisotropy conditions for the lowest Hamiltonians

$$f_{111} = -p_1^3 + p_3 - up_1 \quad \text{and} \quad f_{113} = -p_1^2 p_3 + p_5 - vp_3 - wp_1.$$

One gets

$$\begin{aligned} \{f_{111}, f_{113}\} = & (-u_{x_2} + 3w_{x_1}) f_{111} + (-2u_{x_1} + 3v_{x_1}) f_{113} + (-2u_{x_1} + 3v_{x_1}) p_5 \\ & + (-u_{x_3} + 3w_{x_1} + 2vu_{x_1} - 3vv_{x_1} - v_{x_3} + uv_{x_1}) p_3 \\ & + (uu_{x_3} - 2uw_{x_1} + 2wu_{x_1} - 3wv_{x_1} - wu_{x_1} - u_{x_2}v + u_{x_5} - w_{x_3}) p_1. \end{aligned} \quad (4.7)$$

The r.h.s. of (4.7) vanishes on  $\Gamma$  if

$$\begin{aligned} 2u_{x_1} - 3v_{x_1} &= 0, \\ u_{x_3} - 3w_{x_1} + v_{x_3} - uv_{x_1} &= 0, \\ u_{x_5} + uu_{x_3} - 2uw_{x_1} - u_{x_1}w - vu_{x_3} - w_{x_3} &= 0. \end{aligned}$$

From the first equation as usual one has  $v = (2/3)u$ . So, one gets the following system

$$\begin{aligned} w_{x_1} + \frac{1}{9}(u)_{x_1}^2 - \frac{5}{9}u_{x_3} &= 0, \\ u_{x_5} - \frac{7}{9}uu_{x_3} + \frac{4}{9}u^2u_{x_1} - u_{x_1}w - w_{x_3} &= 0 \end{aligned}$$

or, equivalently, the equation

$$\frac{9}{5}u_{x_5} - uu_{x_3} + u^2u_{x_1} - u_{x_1}\partial_{x_1}^{-1}u_{x_3} - \partial_{x_1}^{-1}u_{x_3x_3} = 0. \quad (4.8)$$

It is just the dKP equation of the B type [27, 15]. In a similar manner one can get the whole dBKP hierarchy which represent itself the coisotropic deformations of the Jordan's triple systems (4.3).

The form of the structure constants analogous to (2.1) of dKP case is the following

$$\begin{aligned} C_{ikl}^m = & \delta_{i+k+l+1}^m + H_{2(k+l-m)+1}^{2i+1} + H_{2(i+l-m)+1}^{2k+1} + H_{2(i+k-m)+1}^{2l+1} \\ & + \sum_{\substack{p,t=0 \\ p+t=l-1-m; m \geq 0}}^{\infty} H_{2p+1}^{2i+1} H_{2t+1}^{2k+1} + \sum_{\substack{p,s=0 \\ p+s=k-1-m; m \geq 0}}^{\infty} H_{2p+1}^{2i+1} H_{2s+1}^{2l+1} \\ & + \sum_{\substack{t,s=0 \\ t+s=i-1-m; m \geq 0}}^{\infty} H_{2t+1}^{2k+1} H_{2s+1}^{2l+1} \end{aligned} \quad (4.9)$$

where  $H_{2p+1}^{2i+1} = 0$  at  $p \leq 1$ . The associativity conditions (4.5) are equivalent to infinite set of cubic equations for  $H_{2p+1}^{2i+1}$  and the coisotropy conditions are equivalent to these algebraic associativity equations and the exactness equations similar to the dKP case.

The two point dKP hierarchy with the involution  $p_i \rightarrow -q_i$ ,  $q_i \rightarrow -p_i$  provides us with one more interesting example. In this case one has two families of relations (4.1), i.e.

$$\begin{aligned} p_{2i+1} &= \sum_{k=0}^i v_{i,k}(x, y) p_1^{2k+1}, \\ q_{2i+1} &= \sum_{k=0}^i w_{i,k}(x, y) q_1^{2k+1} \end{aligned} \quad (4.10)$$

glued by

$$p_1 q_1 = u(x, y) \quad (4.11)$$

where  $(x, y) = (x_1, x_3, x_5, \dots; y_1, y_3, y_5, \dots)$ .

The corresponding closed “algebra” is defined by the following table of multiplication

$$\begin{aligned} p_i p_k p_l &= \sum_{m=1} A_{ikl}^m(x, y) p_m, \\ q_i q_k q_l &= \sum_{m=1} B_{ikl}^m(x, y) q_m, \\ p_i p_k q_l &= \sum_{m=1} C_{ikl}^m(x, y) p_m + \sum_{m=1} D_{ikl}^m(x, y) q_m, \\ p_i q_k q_l &= \sum_{m=1} E_{ikl}^m(x, y) p_m + \sum_{m=1} F_{ikl}^m(x, y) q_m. \end{aligned} \quad (4.12)$$

The formulas (4.12) define the Jordan’s double triple system. In the parametrization (4.10), (4.11) the structure constants  $A$  and  $B$  are given by the relations of the type (4.2) while

$$\begin{aligned} C_{111}^1 &= u, & D_{111}^m &= 0, & E_{111}^m &= 0, & F_{111}^1 &= u, \\ C_{113}^1 &= w_{1,1} u, & D_{113}^1 &= u^2, & E_{311}^1 &= u^2, & F_{311}^1 &= uv_{1,1}, \end{aligned} \quad (4.13)$$

and so on.

In general, the “structure constants”  $A, B, C, D, E, F$  obey the set of cubic “associativity” conditions. The coisotropy condition for the Hamiltonians defined by (4.12) again give rise to the integrable deformations of the triple system (4.12). It can be called the two-point dBKP hierarchy.

Let us consider the set of three lowest Hamiltonians from (4.10), (4.11), i.e.

$$\begin{aligned} f &= p_3 - p_1^3 + vp, \\ \tilde{f} &= q_3 - q_1^3 + wq_1, \\ H &= p_1 q_1 - u(x, y). \end{aligned} \quad (4.14)$$



The coisotropy conditions for the zero set  $\Gamma : f = g = H = 0$  take the form

$$\begin{aligned}\{f, \tilde{f}\} &= 3(w_{x_1}p_1 - v_{y_1}q_1)H, \\ \{f, H\} &= v_{x_1}H, \\ \{\tilde{f}, H\} &= w_{y_1}H\end{aligned}\tag{4.15}$$

together with the equations

$$\begin{aligned}u_{x_3} + (uv)_{x_1} &= 0, \\ v_{y_1} - 3u_{x_1} &= 0, \\ u_{y_3} + (uw)_{y_1} &= 0, \\ w_{x_1} - 3u_{y_1} &= 0.\end{aligned}\tag{4.16}$$

This system represent the dispersionless limit of the Nizhnik-Veselov-Novikov (dNVN) equation [16] (more precisely for solutions of the form  $u = u(x_1, y_1, x_3 + y_3)$ ). The whole family of the coisotropic deformations for the algebra (4.12) is given by the dNVN hierarchy.

Central systems for the algebraic structures discussed above, existence of tau-functions and corresponding dispersionless Hirota's equations will be considered elsewhere.

## 5 Stationary reductions of dispersionless hierarchies and dynamical systems integrable on a fixed energy level.

Integrable hierarchies and the whole construction considered in this paper admit reductions for which some variables  $x_i$  are cyclic one. For example, for the dKP case the cyclicity of the variable  $x_2$  reduces it to the dKdV hierarchy. Under such a reduction  $\{p_i, f\} = 0$  for any function  $f$  and hence  $p_i = \text{const}$ . So the number of canonical variables is reduced effectively by two and the number of Hamiltonians becomes equal to the one-half of total number of variables.

Let us analyze such a situation more carefully. We begin with the dKP equation and assume that the variable  $x_3$  is cyclic. Hence,  $p_3 = \text{const}$ . For this stationary reduction the dKP equation becomes

$$\frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 (u^2)}{\partial x_1^2} = 0\tag{5.1}$$

and one has two Hamiltonians (1.10)

$$\begin{aligned}H &:= h_2 = -p_2 + p_1^2 + u(x_1, x_2), \\ H_1 &:= h_3 = p_1^3 + \frac{3}{2}up_1 + v + \text{const}\end{aligned}\tag{5.2}$$

where  $\frac{\partial v}{\partial x_1} = \frac{4}{3} \frac{\partial u}{\partial x_2}$ . These Hamiltonians are in involution  $\{H, H_1\} = 0$ .

For the Hamiltonian  $H$  the associated dynamical system contains, in particular, the equations

$$\begin{aligned} \frac{\partial x_1}{\partial t} &= 2 \frac{\partial H}{\partial p_1} = 2p_1, \\ \frac{\partial p_1}{\partial t} &= -\frac{\partial H}{\partial x_1} = -\frac{\partial u}{\partial x_1}, \end{aligned} \quad (5.3)$$

and, hence,

$$\frac{\partial^2 x_1}{\partial t^2} = -2 \frac{\partial u}{\partial x_1}. \quad (5.4)$$

For the dynamical system (5.3) the function  $H_1$  is the integral of motion cubic in momentum. This observation reproduces the result of the paper [28] that the dynamical system (5.3) and (5.4) with the potential  $u(x_1, x_2)$  which obey equation (5.1) has the additional integral of motion cubic in  $p_1$ .

In the case of cyclicity with respect to the variable  $x_n$  ( $n \geq 4$ ) one has the Hamiltonian  $H_n = p_1^n + \dots$ , in involution with  $H$ , which is the integral of motion for the system (5.3) of the order  $n$  (see again [28]).

Another simple example is provided by the stationary dNVN system (4.14), (4.15). In order to establish the connection with the standard two-dimensional dynamical systems we impose the constraints  $y_1 = \bar{x}_1$ ,  $q_1 = \bar{p}_1$ ,  $q_3 = \bar{p}_3$ ,  $w = \bar{v}$  where bar means a complex conjugation, and assume that  $u$  depends only on  $x_1$ ,  $\bar{x}_1$  and  $t = x_3 + y_3$ . Passing to new variables  $x_1 \rightarrow z := x_1 + ix_2$ ,  $p_1 \rightarrow p := p_1 + ip_2$ , one rewrites the Hamiltonian  $H$  in the form

$$H = p_1^2 + p_2^2 + u(x_1, x_2, t) \quad (5.5)$$

while the corresponding equation looks like

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial(uv)}{\partial z} + \frac{\partial(u\bar{v})}{\partial \bar{z}} &= 0, \\ \frac{\partial v}{\partial \bar{z}} &= 3 \frac{\partial u}{\partial z}. \end{aligned} \quad (5.6)$$

It is the dispersionless Veselov-Novikov (dVN) equation [2, 16]. Instead of two Hamiltonians  $f$  and  $\tilde{f}$  (4.14) one has for equation (5.6) their sum

$$\tilde{H} = f + \tilde{f} = p_3 + \bar{p}_3 - p^3 + vp - \bar{p}^3 + \bar{v}\bar{p} \quad (5.7)$$

and

$$\{H, \tilde{H}\} = (v_z + \bar{v}_{\bar{z}}) H. \quad (5.8)$$

The formulae (5.7-5.8) mean that for any solution of the dVN equation (5.6) the dynamical system with the Hamiltonian  $H$  (5.5) possesses on the level of zero energy  $E = 0$  the additional integral of motion  $\tilde{H}$  (5.7). In particular, in the stationary case  $\frac{\partial u}{\partial t} = 0$ , then  $p_3 + \bar{p}_3$  is a constant and hence

$$\tilde{H} = -p^3 + vp - \bar{p}^3 + \bar{v}\bar{p} + \text{const}. \quad (5.9)$$

So, in this case one has the dynamical system with two degrees of freedom which has at the zero energy level  $E = 0$  the additional cubic integral of motion (5.9) if the potential  $u$  is a solution of the stationary  $dVN$  equation

$$\frac{\partial(uv)}{\partial z} + \frac{\partial(u\bar{v})}{\partial \bar{z}} = 0, \quad \frac{\partial v}{\partial \bar{z}} = 3 \frac{\partial u}{\partial z}. \quad (5.10)$$

Such integrals of motion are usually called conditional or configurational integrals of motion (see e.g. [29, 30]). Within a different approach the connection between the existence of the conditional cubic integral (5.9) and equation (5.10) has been established in [31] (formula (7.5.12)). Similarly the higher odd order conditional integrals for the Hamiltonian system (5.5) found in [31] are the Hamiltonians  $H_n$  associated with higher stationary  $dVN$  equations.

So, stationary dKP, stationary  $dVN$  hierarchies and other stationary dispersionless integrable hierarchies provide us with the dynamical systems completely Liouville integrable on the fixed energy level.

Full dispersionless hierarchies in our scheme are characterized by the existence of number of independent Hamiltonians which is equal to the one-half of dimension of the symplectic space minus one. They represent the cases closest to the Liouville completely integrable ones and one may call them next to the Liouville completely integrable.

**Acknowledgements.** The authors are grateful to A. Moro for the help in preparation of the paper. The work has been partly supported by the grants COFIN 2004 "Sintesi", and COFIN 2004 "Nonlinear Waves and Integrable systems"

## References

- [1] Zakharov V.E.: *Benney equations and quasi-classical approximation in the inverse problem method*, Func. Anal. Priloz. 14, 89-98 (1980).
- [2] Krichever I.M.: *Averaging method for two-dimensional integrable equations*, Func. Anal. Priloz. 22, 37-52 (1988).
- [3] Kodama Y.: *A method for solving the dispersionless KP equation and its exact solutions*, Phys. Lett. A 129, 2223-2226 (1988).
- [4] Dubrovin B.A. and Novikov S.P.: *Hydrodynamics of weakly deformed soliton lattices : differential geometry and Hamiltonian theory*, Russian Math. Surveys 44, 35-124 (1989).
- [5] Kodama Y.: *Solutions of the dispersionless Toda equation*, Phys. Lett. A 147, 447-482 (1990).
- [6] Kuperschmidt B.A.: *The quasiclassical limit of the modified KP hierarchy*, J. Phys. A : Math. Phys. 23, 871-886 (1990).

- [7] Takasaki K. and Takebe T.: *Sdiff(2) KP hierarchy*, Int. J. Mod. Phys. A 7, 889-922 (1992).
- [8] Takasaki K. and Takebe T.: *Integrable hierarchies and dispersionless limit*, Rev. Math. Phys. 7, 743-818 (1995).
- [9] Krichever I.M.: *The tau-function of the universal Whitham hierarchy, Matrix models and Topological field theory*, Commun. Pure Appl. Math. 47, 437-475 (1994).
- [10] Zakharov V.E.: *Dispersionless limit of integrable systems in 2+1-dimensions*, in *Singular limits of dispersive waves*, (Ercolani N.M. et al., Eds.), Plenum, New York, 1994.
- [11] Carroll R. and Kodama Y.: *Solutions of the dispersionless Hirota equations*, J. Phys. A: Math. Gen. 28, 6373-6387 (1995).
- [12] Gibbons J. and Tsarev S.P., *Reductions of the Benney equations*, Phys. Lett. A 211, 19-23 (1996).
- [13] Wiegmann P.B. and Zabrodin A.: *Conformal maps and integrable hierarchies*, Commun. Math. Phys. 213, 523-538 (2000).
- [14] Boyaskj A., Marshakov A., Ruchayskiy O., Wiegmann P.B. and Zabrodin A.: *Associativity equations in dispersionless integrable hierarchies*, Phys. Lett. 515, 483-492 (2001).
- [15] Konopelchenko B.G. and Martinez Alonso L.: *Dispersionless scalar hierarchies, Whitham hierarchy and the quasi-classical  $\bar{\partial}$ -method*, J. Math. Phys. 43, 3807-3823 (2003).
- [16] Konopelchenko B.G. and Martinez Alonso L.: *Nonlinear dynamics in the plane and integrable hierarchies of infinitesimal deformations*, Stud. Appl. Math. 109, 313-336 (2002).
- [17] Konopelchenko B.G. and Magri F., *Coisotropic deformations of associative algebras and dispersionless integrable hierarchies*, arXiv:nlin.SI/0606069(2006).
- [18] Weinstein A.: *Coisotropic calculus and Poisson groupoids*, J. Math. Soc. Japan 40, 705-727 (1988).
- [19] Berndt R.: *An introduction to symplectic geometry*, AMS, Providence, 2001.
- [20] Falqui G., Magri F. and Pedroni M.: *Bihamiltonian geometry, Darboux covering and linearization of the KP hierarchy*, Commun. Math. Phys. 197, 303-324, (1998).
- [21] Witten E.: *On the structure of topological phase of two-dimensional gravity*, Nucl. Phys. B 340, 281-332, (1990).

- [22] Dijkgraaf R., Verlinde H. and Verlinde E.: *Topological strings in  $d=1$* , Nucl. Phys. B 352, 59-86, (1991).
- [23] Li L.C.: *Classical  $r$ -matrices and compatible Poisson structures for Lax equations on Poisson algebras*, Commun. Math. Phys. 203,573-592 (1999).
- [24] Blaszkak M.: *Classical  $R$ -matrices on Poisson algebras and related dispersionless systems*, Phys. Lett. A 297, 191-195, (2002).
- [25] Jacobson N.: *Structure and representation of Jordan algebras*, Providence R.I., 1968.
- [26] Neher E.: *Jordan triple systems by the grid approach*, Lect. Notes Math., vol. 1280, Springer-Verlag, Berlin, 1987.
- [27] Takasaki K.: *Quasi-classical limit of BKP hierarchy and  $W$ -infinity symmetries*, Lett. Math. Phys. 28, 177-185 (1993).
- [28] Kozlov V.V.: *On polynomial integrals for dynamical systems with one and half degrees of freedom*, Mat. Zametki 45, 46-51 (1989).
- [29] Birkhoff G.D.: *Dynamical systems*, Amer. Math. Soc., Providence, 1927.
- [30] Whittaker E.T.: *A treatise on the analytical dynamics of particles and rigid bodies*, Cambridge Univ. Press., London, 1937.
- [31] Hietarinta J.: *Direct methods for the search of the second invariant*, Phys. Reports 147, 87-154 (1987).